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Percolation, Fractals, and Anomalous Diffusion

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Both the infinite cluster and its backbone are self-similar at the percolation threshold, p_c. This self-similarity also holds at concentrations p near p_c, for length scales L which are smaller than the percolation connectedness length, ξ. For $L < \xi$, the number of bonds on the infinite cluster scales as L^D, where the fractal dimensionality D is equal to $(d - \beta/\nu)$. Geometrical fractal models, which imitate the backbone and on which physical models are exactly solvable, are presented. Above six dimensions, one has $D = 4$ and an additional scaling length must be included. The effects of the geometrical structure of the backbone on magnetic spin correlations and on diffusion at percolation are also discussed.

KEY WORDS: Percolation theory; fractal dimensionality; self-similarity; fractal model for percolation; percolation above six dimensions; magnetic correlations at percolation; anomalous diffusion at percolation.

1. INTRODUCTION

Many structures in physics are invariant under the change in length scale. In particular, *self-similarity* appears in the shapes of droplets at the critical temperature of continuous phase transitions, in the shapes of the routes covered by random walks, in the shapes of long polymers (i.e., self-avoiding random walks), etc.

As explained by Mandelbrot,[1,2] it is very convenient to describe self-similar structures using the notion of the *fractal dimensionality*, D. When the length scale is multiplied by a factor b, the number of basic units in the structure (e.g., lattice sites, bonds, plaquettes, etc.) is multiplied by a factor b^D.

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The aim of this paper is to review some of the applications of these ideas to the *theory of percolation.* In Section 2 we derive the fractal dimensionality of percolation clusters, and describe its measurements via Monte Carlo simulations. Section 3 then describes a family of fractal structures, which may be used to model the backbone of the infinite cluster at percolation. These simple ideas must be modified above six dimensions, and a generalization is reviewed in Section 4. Section 5 discusses the propagation of magnetic spin correlations at percolation, and Section 6 mentions anomalous discussion and concludes the paper.

2. SELF-SIMILARITY OF PERCOLATION CLUSTERS

Consider a d-dimensional (hypercubic) lattice, in which sites are occupied (or empty) with probability p (or $q = 1 - p$). For small p, one finds only small finite clusters of connected occupied sites (sites are connected via nearest-neighbor bonds). The typical linear size of these clusters grows with p, and diverges at the *percolation threshold*, p_c, as $\xi_p \propto |p - p_c|^{-\nu}$. For $p > p_c$ there appears an *infinite cluster*, i.e., a cluster which connects the opposite boundaries of the system even when its size becomes infinite. The probability per site to belong to this cluster vanishes at p_c, as $P_\infty \propto (p - p_c)^\beta$.

Pictures of the infinite cluster at p_c look self-similar.[3] The pictures contain empty areas of all sizes (down to the microscopic lattice distance, a), and look the same when coarse-grained into cells of size $b \times b$ and reduced in size by a factor b. It has therefore been stated[4,5] that these clusters are *self-similar*, or *fractal.* This self-similarity disappears for $p > p_c$, on length scales larger than ξ_p, when the pictures look *homogeneous.* The same self-similarity is observed for finite clusters (both above and below p_c), on length scales shorter than ξ_p.

A quantitative check of these statements may be performed as follows: Given a point on the infinite cluster, consider the number $M(L)$ of points on the same cluster within a volume L^d (of linear size L) centered at that point. Self similarity implies that

$$M(L) \propto L^D \tag{1}$$

We have recently[6] studied two-dimensional ($d = 2$) Monte Carlo simulations of site percolation, with $p > p_c = 0.5927$. Plots of $\ln M$ versus $\ln L$ indeed exhibit the power law decay (1) with $D = 1.900 \pm 0.009$. This decay appears up to scales $\xi_p \propto (p - p_c)^{-\nu}$, with $\nu = 1.33 \pm 0.08$. For $L > \xi$ we find that $M(L) \propto L^d P_\infty$, and $P_\infty \propto \xi_p^{-\beta/\nu} \propto (p - p_c)^\beta$, with $\beta = 0.140 \pm 0.007$.

In order to interpret these results, consider the conditional probability

$\rho(r)$ that a point at a distance r from the origin (which belongs to the infinite cluster) will also belong to the infinite cluster. If scaling holds, then (for $r \gg a$) the only relevant length is ξ, and we expect the scaling form

$$\rho(r) = P_\infty(p) f(r/\xi) \tag{2}$$

The prefactor, $P_\infty(p)$, represents the expectation that the two sites (at r and at the origin) are uncorrelated. For $r \gg \xi$, when we expect that $\rho(r) \to P_\infty(p)$, i.e., that $f(x)$ approaches a constant as $x \to \infty$. For $r \ll \xi$ we expect $\rho(r)$ to be independent of ξ. This can be achieved only if $f(x) \sim x^{-\beta/\nu}$ for $x \ll 1$. We thus predict that $\rho(r) \propto r^{-\beta/\nu}$ for $r \ll \xi$. The "mass" $M(L)$ is found via $M(L) = \int_0^L d^d r \rho(r)$, and one easily checks that $M(L) \propto L^{d-\beta/\nu}$ for $L < \xi$, yielding Eq. (1), with

$$D = d - \beta/\nu \tag{3}$$

For larger scales, $L > \xi$, we find $M(L) \propto (L/\xi)^d \xi^{d-\beta/\nu}$. The average density $\bar{\rho}(L)$ is thus found to behave as $L^{-\beta/\nu}$ for $L < \xi$ and as $P_\infty(p) \propto \xi^{-\beta/\nu}$ for $L > \xi$. Note that for $L < \xi$ one has $\bar{\rho}(L) \propto \rho(L)$. This can hold only if the infinite cluster is highly correlated, containing "holes" at *all* length scales. Indeed, our measured values of D, β, and ν agree excellently well with Eq. (3).

Similar studies can be performed on the *backbone* of the infinite cluster, in which all the "dead ends" (finite sections of the infinite cluster which can be cut off by cutting a single bond, and thus do not contribute to the propagation of dc current or of magnetic correlations) have been eliminated. The fractal dimensionality of the backbone at $d = 2$ is $D_B \simeq 1.6$,[3] in agreement with $D_\beta = d - \beta_B/\nu$, where $\beta_B \simeq 0.5$.

3. FRACTAL MODEL FOR THE BACKBONE

In addition to the fractal dimensionality, fractal structures are characterized by many other topological and geometrical parameters. For example, the *order of ramification* R at a point P is the smallest number of interactions which one must cut to isolate an (otherwise arbitrary) bounded subset that surrounds P and falls in the scaling range.[2,8] Monte Carlo simulations suggest[3] that the backbone at p_c is *finitely ramified* but not *quasi-one-dimensional*, i.e., that the minimum value of R, $R_{\min}$, is finite but larger than 2.

We have recently investigated[9] a family of nonrandom fractal lattices which imitate these properties. These fractals are d-dimensional generalizations of the *Sierpinski gasket*: We start with a d-dimensional hypertetrahedron. The midpoints of the edges are then connected, creating $(d+1)$ smaller hypertetrahedra. The volume at the center (bounded by faces of

these new tetrahedra) is then erased. The procedure is repeated on the new tetrahedra, down to the microscopic length scale a. The fractal dimensionality of these structures is

$$D = \ln(d+1)/\ln 2 \tag{4}$$

with values close to those of the real backbone at $d = 1, 2, 3, 4$. We also have $R_{\min} = d + 1$, which is the lowest value possible for fully d-dimensional structures.

The advantage of models like the one presented above is that one can *solve* any physical problem *exactly* on them.[8] Putting resistors r on each of the "microscopic" bonds, we find[9] that the effective resistors at scale La (when those on all smaller scale are eliminated) behave as $r(L) \propto L^{\tilde{\zeta}} r$, with

$$\tilde{\zeta} = \ln\left[(d+3)/(d+1)\right]/\ln 2 \tag{5}$$

Writing the conductivity for $p > p_c$ as $\sigma(p) \propto (p - p_c)^{\mu} \propto \xi^{-\tilde{\mu}}$ with $\tilde{\mu} = \mu/\nu$, we expect that

$$\tilde{\mu} = d - 2 + \tilde{\zeta} \tag{6}$$

Our result for $\tilde{\mu}$ agrees reasonably well with direct estimates on the backbone for $d = 1, 2, 3$.[9]

In addition to the conductivity, one can also solve exactly many spin models on the Sierpinski gaskets. Since the order of ramification is finite, all spin models (with short-range interactions on the scale a) exhibit long-range order only at zero temperature ($T = 0$). We solved various discrete spin models (the Ising model, the s-state Potts model), for Sierpinski gaskets at both $d = 2$ and $d = 3$,[8,10] and found that as $T \to 0$ the thermal correlation length always diverges as

$$\xi \propto \exp\left[A \exp(B/T)\right] \tag{7}$$

A comparison of this result with that of alternative geometrical pictures of the backbone at p_c is given below (Section 5).

A similar solution of spin models with continuous symmetry (e.g., the XY or the Heisenberg model) yields

$$\xi \propto T^{-1/\tilde{\zeta}} \tag{8}$$

Note that although the family of Sierpinski gaskets may not represent a unique model for the infinite cluster, it is still very useful as a test ground for the behavior of various models at percolation.

4. PERCOLATION ABOVE SIX DIMENSIONS

It turns out that the discussion which led to Eq. (3) must be modified for $d > 6$. For $d < 6$, hyperscaling relations show that Eq. (3) is equivalent

to

$$D = (\beta + \gamma)/\nu \tag{9}$$

where γ describes the divergence of the mean squared size of the clusters. Equation (9) also arises if one assumes "*strong self-similarity*," i.e., that the number of sites in a cluster of linear size ξ_p scales as $s(\xi_p) \propto \xi_p^D$. Since $s(\xi_p) \propto (p - p_c)^{-1/\sigma}$, with $1/\sigma = \beta + \gamma$ being the "magnetic" exponent,[7] Eq. (9) follows.[11]

Above six dimensions the critical exponents are known to assume their mean field values, $\beta = 1$, $\gamma = 1$, $\nu = 1/2$. Thus, Eq. (3) predicts that $D = d - 2$, while Eq. (9) yields $D = 4$. We have recently shown that only the latter is correct.[12] The important new ingredient in the theory is *the probability to find vertices at which three bonds meet*, w. This probability turns out to be a "*dangerous irrelevant variable*" for $d > 6$.

An alternative extreme to our fractal model (Section 3) is that of the "*links and nodes*" model.[13,14] In this model, the backbone of the infinite cluster is composed of quasi-one-dimensional links, which connect nodes. At high dimensions, the links are expected to behave as random walks, so that the actual number of sites on a backbone link at scale L is of order $M_B(L) \propto L^2$ (i.e., $D_B = 2$). A fraction of these sites, proportional to w, have "dead ends," or "dangling bonds" attached to them. Using the known distribution of the finite clusters, it can be shown that the average "mass" $M_d(L)$ of such a dangling bond is also of order L^2.[12] Thus,

$$M(L) \propto wM_B(L)M_d(L) \propto wL^4 \tag{10}$$

In the above argument we assumed that the backbone cuts each "dangling bond" only once. In a volume of linear size L, the number of sites on the infinite cluster is $M(L) \propto wL^4$, and the number of sites on the backbone is $M_B(L) \propto L^2$. The number of possible additional "meetings" between them is thus of order $wM(L)M_B(L)$, i.e., w^2L^6, and the density of such vertices is w^2L^{6-d}. (Alternatively, this is the density of "meetings" between the "full" infinite cluster, of mass wL^4, and an "average" dangling bond, of "mass" L^2.) For $d > 6$ we see that this renormalized density decreases with increasing L, and therefore our argument is consistent at large L. The theory must be modified for $d < 6$.

Note also that the density of vertices becomes smaller than unity for $L > L_w$, where $w^2L_w^{6-d} = 1$. This explains the physical meaning of L_w: the vertices are in fact dense "*blobs*" of size L_w, and the geometrical picture used above applies only for $L > L_w$.

The parameter w also arises formally (as the coefficient of the cubic term) in the context of the s-state Potts model.[15] In the limit $s \to 1$, this model describes percolation. For $d > 6$, the critical behavior of the model is

governed by the Gaussian fixed point, and scaling near that point may be used to show that $M(L)$ now depends on both L/ξ_p and ξ_p/L_w,[12]

$$M(L, \xi_p, L_w) = w^{-1}\xi_p^{-2}L^d\tilde{m}\left(\frac{L}{\xi_p}, \frac{\xi_p}{L_w}\right) \tag{11}$$

For $L_w \ll L \ll \xi_p$, this reduces to Eq. (10). For $L \gg \xi_p$, this becomes $M \propto L^d P_\infty \propto L^d/(w\xi_p^2)$. In contrast to the case $d < 6$, the crossover between these two limits does not happen at $L \sim \xi$. Instead, we find a series of crossover lengths, between ξ_p and $L_1 \propto \xi_p^{2/(d-4)}$, at which different physical quantities exhibit crossover.

The backbone of the infinite cluster is thus self-similar, for $d > 6$, only in the range $L_w < L < L_1$. In this range, $D_B = 2$ and the geometry is that of random walks. This is clearly very different from the gasket model, presented in Section 3. A combination of the two models, in which more and more quasi-one-dimensional links are added as d is increased, will certainly yield an improved description.

5. MAGNETIC CORRELATIONS

Consider now the *Ising model on the dilute lattice*, with the Hamiltonian

$$\mathcal{H} = -\sum_{\langle ij \rangle} J_{ij} S_i S_j$$

where $S_i = \pm 1$, and $J_{ij} = J$ (if both the nearest neighbor sites i and j are occupied) or $J_{ij} = 0$ (otherwise). The model has no long-range order at $p < p_c$, and orders at the temperature $T_c(p)$ for $p > p_c$, with $T_c(p) \to 0$ as $p \to p_c$. The percolation point $p = p_c$, $T = 0$ is thus a *multicritical point*. If one writes

$$\xi \propto (p - p_c)^{-\nu} X\left[e^{-2K}/(p - p_c)^{\phi}\right] \tag{12}$$

with $K = J/k_B T$, then the crossover exponent ϕ, obtained from the $s \to 1$ limit of the anisotropic s-state Potts model, turns out to be equal to unity for all d.[16] Thus, the critical line is given by

$$\exp\left[-2J/k_B T_c(p)\right] \propto (p - p_c)^{\phi} = (p - p_c) \tag{13}$$

Consider now the "link and nodes" model. Two spins which are separated by a distance $r < \xi$ are assumed to have only one effective link between them. In the naive model, this link is assumed to be quasi-one-dimensional, composed of $L(r)$ bonds. Decimating these bonds, one can replace them by an effective coupling constant K_L, given by

$$\tanh K_L = (\tanh K)^L \tag{14}$$

For large K, this reduces to $\exp(-2K_L) \simeq L\exp(-2K)$. If we assume that the links meet at nodes which are separated by a distance ξ_p, and that $L(\xi_p) \propto (p - p_c)^{-\phi}$, then Eq. (12) follows.

At $d = 2$, one has $\xi_p \propto (p - p_c)^{-\nu}$, with $\nu = 4/3$. Therefore, the above naive picture and $\phi = 1$ imply that $L(\xi) < \xi$, which is impossible. Moreover, the model assumes that all the "mass" is in L. Since $L(\xi_p) \propto \xi_p^{\phi/\nu}$, this implies that $D = \phi/\nu = 1/\nu$, in strong disagreement with known values for $d < 6$. One therefore needs a modified model. A possible modification, suggested by Stanley and Coniglio,[11] adds within each link "blobs" of spins which are multiply connected. If two spins are connected by two bonds in parallel, then the effective coupling constant between them is $2K$. Since $\tanh 2K \simeq 1 - 2e^{-4K}$, such a factor will not contribute to the renormalized coupling constant K_L at the leading order e^{-2K}. To this order, the power L on the right-hand side of Eq. (14) must be replaced by L_1, i.e., the number of *singly connected* bonds along the link (cutting of such a bond breaks the connection between the ends of the link). The number of these "cutting bonds," L_1, may be much smaller than L. Writing $L_1 \propto (p - p_c)^{-\zeta_1}$, we thus identify $\phi = \zeta_1$. Coniglio[17] recently presented a general proof of the result $\zeta_1 = 1$, in any dimension, confirming that $\phi = 1$.

The above argument is valid only if $e^{-2K_L} \simeq L_1 e^{-2K} \ll 1$. As we approach p_c at a fixed temperature, L_1 increases, $L_1 e^{-2K}$ is no longer small, and the expansion of $\tanh K_L$ is no longer justified. Moreover, the number of bonds within the "blobs" is of order $L \propto \xi_p^{D_B} \gg L_1$. In addition to $L_1 e^{-2K}$, we expect "higher-order" terms, like $L_2 e^{-4K}$. L_2 is the number of pairs of bonds such that the cutting of both breaks the connection between the ends of the link. Writing $L_2 \propto (p - p_c)^{-\zeta_2}$, this term behaves as $e^{-4K}(p - p_c)^{-\zeta_2}$. We expect that $\zeta_2 > \zeta_1$. As soon as $L_2 e^{-2K} > L_1$, i.e.,

$$(p - p_c)^{\zeta_2 - \zeta_1} < e^{-2K} \tag{15}$$

these terms will dominate and the scaling with a single variable, Eq. (12), will have to be modified.[18]

Consider now the spin–spin correlation function $\langle S_0 S_r \rangle$. On length scales $r < \xi_p$, the two spins are within a single "link." The correlation function should, therefore, not depend on ξ_p. In the spirit of the Coniglio model, the function $\langle S_0 S_r \rangle$ is now given by

$$\langle S_0 S_r \rangle = (\tanh K)^{L_1(r)} \simeq e^{-L_1(r)/\xi_1}$$

where $\xi_1 = \frac{1}{2} e^{2K}$ is the one-dimensional Ising correlation length, while $L_1(r)$ is the number of singly connected bonds within a distance r. We now generalize the Stanley–Coniglio picture, by adding the assumption that the geometry is *self-similar* on length scales $r < \xi_p$. Noting that $L_1(\xi_p) \propto \xi_p^{\zeta_1/\nu}$,

we thus write $L_1(r) = A_1 r^{\zeta_1/\nu}$ and obtain

$$\langle S_0 S_r \rangle \simeq \exp(-A_1 r^{\zeta_1/\nu}/\xi_1) \tag{16}$$

Clearly, the structure factor $\mathscr{S}(q)$ obtained from Eq. (16) will be quite different from the usual Lorentzian one; at large q we expect that $\mathscr{S}(q) \propto \xi_p^{-\zeta_1/\nu}/q^{2+\zeta_1/\nu}$.[18]

Once the "blobs" are taken into account, in a self-similar way, the exponent in Eq. (16) may have the additional term $Ar^{\zeta_2/\nu}/\xi_1^2$, which dominates at the intermediate length scales $\xi_1^{\nu_p/(\zeta_2-\zeta_1)} < r < \xi_p$.[18] At $p = p_c$, $\xi_p = \infty$, and this new term dominates for large r, thus affecting the small-q dependence of $\mathscr{S}(q)$. The detailed q dependence of $\mathscr{S}(q)$ at p_c is therefore much more complex than hitherto expected.

If one ignores all these effects, then Eq. (12) implies that at $p = p_c$ one has

$$\xi \propto (e^{-2K})^{-\nu/\phi} \propto e^{2K\nu} \tag{17}$$

However, one is tempted to speculate that the growing role played by the multiply connected bonds, as explained above, may eventually yield a crossover to Eq. (7).

6. CONCLUSION

Due to limitations in space and time, we did not have a chance here to review the recent interesting developments in our understanding of *anomalous diffusion* at percolation.[19] The assumption of self-similarity implies that the mean square distance traveled by a random walker after t steps on the infinite cluster at p_c is given by

$$\langle r^2(t) \rangle_\infty \propto t^{2/(2+\theta)} \tag{18}$$

The exponent $(2 + \theta)$ is sometimes referred to as the fractal dimensionality of the random walk at percolation. It has recently been conjectured[20] that $2 + \theta = 3D/2$, but this remains to be checked in detail.

Many other applications, both directly on the percolation clusters and on Sierpinski gaskets, have recently appeared in the literature. As far as this short review is concerned, it is hoped that the reader will be left with the impression that self-similarity is very useful in discussing phenomena at the percolation threshold, and that there remain many interesting related problems for future study.

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